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Physical interpretation and accuracy of the Kubelka–Munk theory

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Abstract
Kubelka–Munk theory is used extensively in a number of industrial applications including colour matching. The main aim of this paper is the derivation of alternative analytical equations, which are more accurate than correspondent Kubelka–Munk formulae for light scattering materials of a finite optical depth. The derived equations are based on asymptotic radiative transfer theory valid for optically thick turbid media with arbitrary phase functions and single scattering albedos.

1. Introduction

Kubelka–Munk theory (KMT) (Kubelka and Munk 1931, Kubelka 1948) describes optical characteristics (e.g. reflectance, transmittance and absorptance) by a variety of light scattering media including paints, textiles and paper. It is widely used in various industrial applications. The theory was formulated at the beginning of the 20th century, when the radiative transfer theory (RTT) was in a rudimentary state. Modern developments in RTT enable the derivation of Kubelka–Munk parameters from first principles.

In this work, Kubelka–Munk equations are compared with the analytical results of asymptotic radiative transfer theory (Germogenova 1961, van de Hulst 1980). Also new equations for the hemispherical reflectance and transmittance under diffuse light illumination of a light scattering layer are proposed. Their accuracy is studied using the numerical solution of the integro-differential radiative transfer equation for the sparse distribution of spherical scatterers. Parameters of new equations are defined in terms of the optical thickness of a scattering layer τ, the probability of photon absorption β and the asymmetry parameter g. For most of real world materials scattering occurs predominantly at small angles and $q = 1 - g$ is a small parameter of the theory. The same is true for $\beta = 1-\omega_0$, where ω_0 is the single scattering albedo defined as the ratio of scattering $K_{\rm sca}$ and extinction $K_{\rm ext}$ coefficients of a turbid layer. The small values of β correspond to the case $K_{\rm abs} \ll K_{\rm ext}$, where $K_{\rm abs}$ is the absorption coefficient.

The next section is devoted to the introduction of main notions and equations of asymptotic radiative transfer theory. We consider the relationship of the asymptotic theory with the results of the Kubelka–Munk two-flux analysis in section 3. Section 4 is devoted to comparisons of equations derived with exact radiative transfer calculations.

2. Theory

2.1. Reflection and transmission functions

Reflection $R_A(\mu_0, \mu, \phi)$ and transmission $T_A(\mu_0, \mu)$ functions of optically thick light scattering layers over a Lambertian surface with the albedo A can be presented in the following form (Germogenova 1961, van de Hulst 1980, Kokhanovsky 2006):

$$R_A(\mu_0, \mu, \phi) = R(\mu_0, \mu, \phi) + \frac{At_{\rm d}(\mu_0)t_{\rm d}(\mu)}{1 - Ar}, \quad (1)$$

$$T_A(\mu_0, \mu) = T(\mu_0, \mu) + \frac{At_{\rm d}(\mu_0)r_{\rm d}(\mu)}{1 - Ar}, \quad (2)$$

where all quantities not having the subscript A are defined for the case of $A = 0$. Also it follows that

$$R(\mu_0, \mu, \phi) = R_\infty(\mu_0, \mu, \phi) - T(\mu_0, \mu)l{\rm e}^{-k\tau}, \quad (3)$$

$$T(\mu_0, \mu) = \frac{m{\rm e}^{-k\tau}}{1 - l^2{\rm e}^{-2k\tau}}u(\mu_0)u(\mu). \quad (4)$$

Here the pair (μ_0, μ) gives the cosines of the zenith incidence and observation angles, respectively, ϕ is the relative azimuth, $\tau = K_{\rm ext}L$ is the optical thickness of a scattering plane-parallel layer, L is the geometrical thickness, $t_{\rm d}(\mu_0)$ is the

diffuse transmittance of a layer under illumination along the direction having the angle $\vartheta_0 = \arccos(\mu_0)$ with respect to the normal to a layer, r is the diffuse reflectance of a turbid layer under the diffuse illumination conditions (the spherical albedo), $r_{\mathrm{d}}(\mu)$ is the reflectance along the direction specified by viewing the zenith angle $\vartheta = \arccos(\mu)$ for the diffuse illumination conditions (the plane albedo) and $R_\infty(\mu_0, \mu, \phi)$ is the reflection function of a semi-infinite scattering layer having the same local optical characteristics (e.g. the single scattering albedo ω_0 and the phase function $p(\theta)$, θ is the scattering angle) as a finite layer under study. The functions $r_{\mathrm{d}}(\mu)$, $t_{\mathrm{d}}(\mu)$ and the spherical albedo r are defined as

$$r_{\mathrm{d}}(\mu) = 2\int_0^1 \bar{R}(\mu_0, \mu)\mu_0\,\mathrm{d}\mu_0,$$

$$t_{\mathrm{d}}(\mu) = 2\int_0^1 \bar{T}(\mu_0, \mu)\mu_0\,\mathrm{d}\mu_0, \tag{5}$$

$$r = 2\int_0^1 r_{\mathrm{d}}(\mu_0)\mu_0\mathrm{d}\mu_0,$$

where $\bar{R}(\mu_0, \mu) = \frac{1}{2\pi}\int_0^{2\pi} R(\mu_0, \mu, \phi)\,\mathrm{d}\phi$ and $\bar{T}(\mu_0, \mu) = \frac{1}{2\pi}\int_0^{2\pi} R(\mu_0, \mu, \phi)\,\mathrm{d}\phi$. The constants (k, l, m) and the escape function $u(\mu)$ do not depend on τ and can be obtained from the solution of integral equations as described by van de Hulst (1980) and Kokhanovsky (2006). In particular, it follows that

$$l = 2\int_0^1 i(-\xi)u(\xi)\xi\,\mathrm{d}\xi \qquad m = 2\int_{-1}^1 i^2(\xi)\xi\,\mathrm{d}\xi, \tag{6}$$

$$(1 - k\xi)i(\xi) = \frac{1-\beta}{2}\int_{-1}^1 \bar{p}(\xi, \eta)i(\eta)\,\mathrm{d}\eta. \tag{7}$$

Equation (7) determines the light field $i(\xi)$ in the deep layers of a turbid medium. Here ξ is the cosine of the zenith observation angle with respect to the downward normal to a scattering layer and $\bar{p}(\xi, \eta)$ is the azimuthally averaged phase function of a scattering layer. The angular distribution of the scattered light $i(\xi)$ is preserved in the deep layers. The downward propagating flux attenuates proportionally to $\exp(-k\tau^*)$, where τ^* is the optical depth inside the scattering layer counting from its top. The light flux attenuates with the diffusion exponent k, which can be found from equation (7).

The escape function is obtained from the solution of the Milne problem (van de Hulst 1980). This function describes the angular distribution of light escaping a semi-infinite light scattering layer with sources placed at the infinite depth. Alternatively, it can be derived from the following equation:

$$u(\eta) = 2m^{-1}\int_0^1 \bar{R}_\infty(\eta, \eta')i(-\eta')\eta'\,\mathrm{d}\eta' - m^{-1}i(\eta). \tag{8}$$

2.2. *Derivation of main equations*

Expressions for $r_{\mathrm{d}A}(\mu)$, $t_{\mathrm{d}A}(\mu)$ and also for the spherical albedo r_A (A signifies that the equations are valid for arbitrary Lambertian underlying surface albedo A) can be derived from equations (1) and (2) accounting for definitions (5). Namely, one derives

$$r_{\mathrm{d}A}(\mu) = r_{\mathrm{d}}(\mu) + \frac{At^2u(\mu)}{n(1 - Ar_{\mathrm{s}})},$$
$$r_{\mathrm{d}}(\mu) = r_{\mathrm{d}\infty}(\mu) - tln^{-1}u(\mu)\mathrm{e}^{-k\tau}, \tag{9}$$

$$r_A = r + \frac{At^2}{1 - Ar}, \qquad r = r_\infty - lt\mathrm{e}^{-k\tau}, \tag{10}$$

$$t_{\mathrm{d}A}(\mu) = t_{\mathrm{d}}(\mu) + \frac{Atr_{\mathrm{d}}(\mu)}{1 - Ar_{\mathrm{s}}}, \qquad t_{\mathrm{d}}(\mu) = tn^{-1}u(\mu), \tag{11}$$

where $r_\infty \equiv r(\tau \to \infty)$ and t is the global transmittance defined as

$$t \equiv 2\int_0^1 t_{\mathrm{d}}(\mu)\mu\,\mathrm{d}\mu = \frac{mn^2\mathrm{e}^{-k\tau}}{1 - l^2\mathrm{e}^{-2k\tau}} \tag{12}$$

and

$$n = 2\int_0^1 u(\mu)\mu\,\mathrm{d}\mu. \tag{13}$$

Also it follows that

$$t_A = \frac{t}{1 - Ar}. \tag{14}$$

The radiative characteristics $r_{\mathrm{d}}(\mu)$, $t_{\mathrm{d}}(\mu)$, r_∞, t and also $R(\mu_0, \mu, \phi)$ and $T(\mu_0, \mu)$ can be directly measured. Therefore, equations shown above are of importance for the interpretation of corresponding experiments. We summarize the main equations in table 1.

For the use of analytical equations given in table 1, one needs to calculate the parameters (k, l, m, n, r_∞) and also the functions $u(\mu)$, $R_\infty(\mu_0, \mu, \phi)$, $r_{\mathrm{d}\infty}(\mu)$. It was found (van de Hulst 1974, King and Harshvardan 1986) that parameters (k, l, m, n, r_∞) can be approximated as follows:

$$k = \left(\sqrt{3}s - \frac{(0.985 - 0.253s)s^2}{6.464 - 5.464s}\right)(1 - \omega_0 g), \tag{15}$$

$$l = \frac{(1-s)(1 - 0.681s)}{1 + 0.729s}, \tag{16}$$

$$m = (1 + 1.537s)\ln\left(\frac{1 + 1.8s - 7.087s^2 + 4.74s^3}{(1 - 0.819s)(1-s)^2}\right), \tag{17}$$

$$n = \sqrt{\frac{(1-s)(1 + 0.414s)}{1 + 1.888s}}, \tag{18}$$

$$r_\infty = \frac{(1-s)(1 - 0.139s)}{1 + 1.17s}, \tag{19}$$

where

$$s = \sqrt{\frac{1 - \omega_0}{1 - \omega_0 g}} \tag{20}$$

is the similarity parameter and

$$g = \frac{1}{4}\int_0^\pi p(\theta)\sin(2\theta)\,\mathrm{d}\theta \tag{21}$$

is the asymmetry parameter.

Functions $u(\mu)$, $R_\infty(\mu_0, \mu, \phi)$, $r_{\mathrm{d}\infty}(\mu)$ cannot be parameterized in terms of the single parameter s. An important point is that the function $u(\mu)$ only weakly depends on the asymmetry parameter. So one can use the look-up table for this

Table 1. Radiative transfer characteristics of optically thick layers.

Radiative characteristic	$A = 0$	$A \neq 0$
Transmission function $T(\mu_0, \mu)$	$tn^{-2}u(\mu_0)u(\mu)$	$tn^{-2}u(\mu_0)u(\mu) + \dfrac{Atn^{-1}u(\mu_0)r_d(\mu)}{1 - Ar_s}$
Diffuse transmittance $t_d(\mu)$	$tn^{-1}u(\mu)$	$tn^{-1}u(\mu) + \dfrac{Atr_d(\mu)}{1 - Ar_s}$
Global transmittance t	$\dfrac{mn^2e^{-k\tau}}{1 - l^2e^{-2k\tau}}$	$\dfrac{mn^2e^{-k\tau}}{(1 - l^2e^{-2k\tau})(1 - Ar_s)}$
Reflection function $R_\infty(\mu_0, \mu, \phi)$	$R_\infty(\mu_0, \mu, \phi) - ltn^{-2}u(\mu_0)u(\mu)e^{-k\tau}$	$R_\infty(\mu_0, \mu, \phi) - ltn^{-2}u(\mu_0)u(\mu)e^{-k\tau} + \dfrac{At^2n^{-2}u(\mu_0)u(\mu)}{1 - Ar_s}$
Diffuse reflectance (plane albedo) $r_d(\mu)$	$r_{d\infty}(\mu) - ltn^{-1}u(\mu)e^{-k\tau}$	$r_{d\infty}(\mu) - ltn^{-1}u(\mu) + \dfrac{At^2n^{-1}u(\mu)}{1 - Ar_s}$
Spherical albedo r_s	$r_\infty - lte^{-k\tau}$	$r_\infty - lte^{-k\tau} + \dfrac{At^2}{1 - Ar_s}$

Table 2. The spherical albedo r_s as the function of optical thickness τ calculated using the exact radiative transfer code SCIATRAN (Rozanov *et al* 2005) for the phase function of spherical polydispersions of water droplets as described in the text ($g = 0.85$) and several values of the probability of photon absorption β.

τ	$\beta = 0.001$	$\beta = 0.005$	$\beta = 0.01$	$\beta = 0.05$
0.1000E + 01	0.1320E + 00	0.1300E + 00	0.1276E + 00	0.1099E + 00
0.2000E + 01	0.2171E + 00	0.2122E + 00	0.2063E + 00	0.1663E + 00
0.3000E + 01	0.2832E + 00	0.2749E + 00	0.2651E + 00	0.2025E + 00
0.4000E + 01	0.3373E + 00	0.3254E + 00	0.3115E + 00	0.2269E + 00
0.5000E + 01	0.3828E + 00	0.3672E + 00	0.3490E + 00	0.2440E + 00
0.6000E + 01	0.4219E + 00	0.4023E + 00	0.3800E + 00	0.2561E + 00
0.7000E + 01	0.4558E + 00	0.4323E + 00	0.4058E + 00	0.2647E + 00
0.8000E + 01	0.4856E + 00	0.4581E + 00	0.4274E + 00	0.2709E + 00
0.9000E + 01	0.5120E + 00	0.4805E + 00	0.4458E + 00	0.2754E + 00
0.1000E + 02	0.5354E + 00	0.5000E + 00	0.4613E + 00	0.2786E + 00
0.1100E + 02	0.5565E + 00	0.5171E + 00	0.4746E + 00	0.2810E + 00
0.1200E + 02	0.5755E + 00	0.5322E + 00	0.4860E + 00	0.2827E + 00
0.1300E + 02	0.5927E + 00	0.5454E + 00	0.4958E + 00	0.2839E + 00
0.1400E + 02	0.6082E + 00	0.5572E + 00	0.5043E + 00	0.2848E + 00
0.1500E + 02	0.6224E + 00	0.5677E + 00	0.5115E + 00	0.2855E + 00
0.1600E + 02	0.6354E + 00	0.5769E + 00	0.5178E + 00	0.2860E + 00
0.1700E + 02	0.6473E + 00	0.5853E + 00	0.5233E + 00	0.2864E + 00
0.1800E + 02	0.6584E + 00	0.5927E + 00	0.5280E + 00	0.2866E + 00
0.1900E + 02	0.6685E + 00	0.5994E + 00	0.5321E + 00	0.2868E + 00
0.2000E + 02	0.6779E + 00	0.6054E + 00	0.5356E + 00	0.2869E + 00
0.2100E + 02	0.6866E + 00	0.6107E + 00	0.5387E + 00	0.2870E + 00
0.2200E + 02	0.6946E + 00	0.6156E + 00	0.5414E + 00	0.2871E + 00
0.2300E + 02	0.7021E + 00	0.6198E + 00	0.5437E + 00	0.2872E + 00
0.2400E + 02	0.7090E + 00	0.6238E + 00	0.5457E + 00	0.2872E + 00
0.2500E + 02	0.7156E + 00	0.6273E + 00	0.5475E + 00	0.2872E + 00
0.2600E + 02	0.7216E + 00	0.6304E + 00	0.5490E + 00	0.2873E + 00
0.2700E + 02	0.7273E + 00	0.6333E + 00	0.5504E + 00	0.2873E + 00
0.2800E + 02	0.7327E + 00	0.6358E + 00	0.5515E + 00	0.2873E + 00
0.2900E + 02	0.7377E + 00	0.6382E + 00	0.5525E + 00	0.2873E + 00
0.3000E + 02	0.7424E + 00	0.6403E + 00	0.5534E + 00	0.2873E + 00

function calculated, e.g. using the Heney–Greenstein phase function (van de Hulst 1980) with the asymmetry parameter $g = 0.85$. It follows that $u(\mu, \omega_0 = 1) = 3(1 + 2\mu)/7$ with the error smaller than 2% at $\mu \geqslant 0.2$ (Kokhanovsky 2006). The convenient parametrization for $r_{d\infty}(\mu)$ was proposed by Kokhanovsky and Sokoletsky (2006).

The system of equations shown above can be used to calculate all radiative properties of a thick layer except $R_\infty(\mu_0, \mu, \phi)$. The function $R_\infty(\mu_0, \mu, \phi)$ depends on three angular parameters and also on the single scattering albedo and the phase function. $R_\infty(\mu_0, \mu, \phi)$ can be calculated using the code developed by Mishchenko *et al* (1999). The code is freely available on the Internet (http://www.giss.nasa.gov/~crmim/).

3. The relationship with Kubelka–Munk theory for finite turbid layers

Kubelka–Munk equations for a finite scattering layer over a black underlying surface can be written in the following form

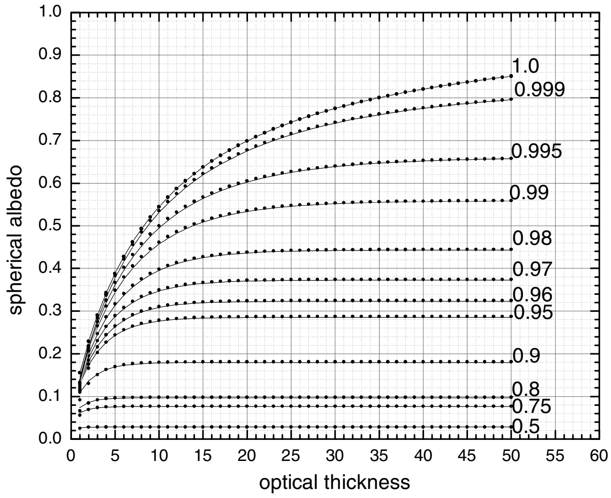

Figure 1. Dependence of the spherical albedo on the optical thickness for different values of ω_0 and for the phase function as specified in the text ($g = 0.85$). Points give results of exact calculations. Lines correspond to the asymptotic theory.

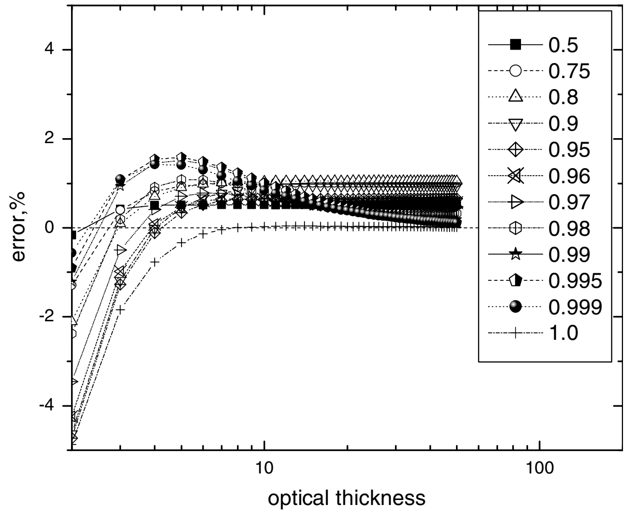

Figure 2. The relative error δ of the spherical albedo calculated using equation (23) for different values of ω_0 and $g = 0.85$ as the function of the optical thickness. Results shown in figure 1 were used for the calculation of δ.

Table 3. The transmittance t as the function of optical thickness τ calculated using the exact radiative transfer code SCIATRAN (Rozanov *et al* 2005) for the phase function of spherical polydispersions of water droplets as described in the text ($g = 0.85$) and several values of the probability of photon absorption β.

τ	$\beta = 0.001$	$\beta = 0.005$	$\beta = 0.01$	$\beta = 0.05$
0.1000E + 01	0.5335E + 00	0.5335E + 00	0.5281E + 00	0.4720E + 00
0.2000E + 01	0.6495E + 00	0.6495E + 00	0.6391E + 00	0.5361E + 00
0.3000E + 01	0.6575E + 00	0.6575E + 00	0.6426E + 00	0.5033E + 00
0.4000E + 01	0.6320E + 00	0.6320E + 00	0.6131E + 00	0.4461E + 00
0.5000E + 01	0.5973E + 00	0.5973E + 00	0.5748E + 00	0.3867E + 00
0.6000E + 01	0.5618E + 00	0.5618E + 00	0.5360E + 00	0.3322E + 00
0.7000E + 01	0.5284E + 00	0.5284E + 00	0.4994E + 00	0.2842E + 00
0.8000E + 01	0.4977E + 00	0.4977E + 00	0.4658E + 00	0.2428E + 00
0.9000E + 01	0.4699E + 00	0.4699E + 00	0.4352E + 00	0.2073E + 00
0.1000E + 02	0.4447E + 00	0.4447E + 00	0.4074E + 00	0.1769E + 00
0.1100E + 02	0.4218E + 00	0.4218E + 00	0.3820E + 00	0.1510E + 00
0.1200E + 02	0.4010E + 00	0.4010E + 00	0.3589E + 00	0.1290E + 00
0.1300E + 02	0.3819E + 00	0.3819E + 00	0.3376E + 00	0.1101E + 00
0.1400E + 02	0.3645E + 00	0.3645E + 00	0.3181E + 00	0.9404E − 01
0.1500E + 02	0.3483E + 00	0.3483E + 00	0.3000E + 00	0.8031E − 01
0.1600E + 02	0.3335E + 00	0.3335E + 00	0.2833E + 00	0.6859E − 01
0.1700E + 02	0.3197E + 00	0.3197E + 00	0.2677E + 00	0.5859E − 01
0.1800E + 02	0.3068E + 00	0.3068E + 00	0.2532E + 00	0.5004E − 01
0.1900E + 02	0.2948E + 00	0.2948E + 00	0.2397E + 00	0.4275E − 01
0.2000E + 02	0.2836E + 00	0.2836E + 00	0.2271E + 00	0.3652E − 01
0.2100E + 02	0.2731E + 00	0.2731E + 00	0.2153E + 00	0.3119E − 01
0.2200E + 02	0.2633E + 00	0.2633E + 00	0.2042E + 00	0.2665E − 01
0.2300E + 02	0.2540E + 00	0.2540E + 00	0.1938E + 00	0.2276E − 01
0.2400E + 02	0.2452E + 00	0.2452E + 00	0.1839E + 00	0.1944E − 01
0.2500E + 02	0.2369E + 00	0.2369E + 00	0.1747E + 00	0.1661E − 01
0.2600E + 02	0.2291E + 00	0.2291E + 00	0.1660E + 00	0.1419E − 01
0.2700E + 02	0.2217E + 00	0.2217E + 00	0.1577E + 00	0.1212E − 01
0.2800E + 02	0.2146E + 00	0.2146E + 00	0.1500E + 00	0.1036E − 01
0.2900E + 02	0.2079E + 00	0.2079E + 00	0.1426E + 00	0.8846E − 02
0.3000E + 02	0.2015E + 00	0.1356E + 00	0.8816E − 01	0.7557E − 02

(Kubelka 1948):

$$t_{\mathrm{KM}} = \frac{(1 - r_\infty^2)\mathrm{e}^{-\bar{x}}}{1 - r_\infty^2 \mathrm{e}^{-2\bar{x}}}, \qquad r_{\mathrm{KM}} = \frac{r_\infty(1 - \mathrm{e}^{-2\bar{x}})}{1 - r_\infty^2 \mathrm{e}^{-2\bar{x}}}, \tag{22}$$

where $\bar{x} = \sigma L$, $\sigma = \sqrt{K(K+S)}$ and the subscript KM signifies that equations are valid under assumptions as introduced by Kubelka and Munk (1931). The last equation can also be written as follows: $r_{\mathrm{KM}} = r_\infty(1 - t\exp(-\bar{x}))$. The main shortcoming of KMT is the fact that diffuse absorption (K) and scattering (S) coefficients have no rigorous electrodynamic meaning. Therefore, their calculation from first principles is hardly possible.

It follows from asymptotic theory developed above after simple algebraic transformations for the case of

$A = 0$ that

$$t = \frac{\alpha e^{-x}}{1 - l^2 e^{-2x}}, \qquad r = \frac{r_\infty (1 - \gamma e^{-2x})}{1 - l^2 e^{-2x}}, \tag{23}$$

where $\alpha = mn^2$, $\gamma = l^2 + mn^2 l r_\infty^{-1}$, $x = k\tau$. Equations (22) and (23) are valid at arbitrary absorption of light in the medium. They can be simplified at $\omega_0 = 1$. In particular, it follows from equations (23) in this particular case (Kokhanovsky 2006):

$$t = (\rho + 0.75\tau(1 - g))^{-1}, \qquad r = 1 - t, \tag{24}$$

where $\rho \approx 1.072$.

Kubelka–Munk equations (22) can be derived from equations (23) under the following assumptions:

$$\bar{x} = x, \qquad \alpha = 1 - r_\infty^2, \qquad l = r_\infty. \tag{25}$$

The first condition means that the parameter σ is nothing other than the product of the extinction coefficient $K_{\rm ext}$ and the diffusion exponent k. Both parameters have a clear physical sense and can easily be calculated or measured. Also KMT indirectly implies that $l^2 + mn^2 = 1$, which is a rigorous result only for nonabsorbing media.

Equations (23) enable understanding the reason behind the successful use of KMT for more than 75 years for industrial applications. Kubelka–Munk equations (22) capture the dependence of light reflectance and transmittance of turbid media on the geometrical thickness of a scattering layer in a correct way. Also all parameters involved can be determined experimentally. We found that for intensely scattering materials ($\omega_0 \to 1$), the conditions $l^2 + mn^2 = 1$, $l = r_\infty$ and $\gamma = 1$ give a good approximation to reality. Then equations (23) are reduced to much simpler forms given by equations (22) with $\sigma = kK_{\rm ext}$.

Rigorous equations (23) look more complicated than corresponding Kubelka–Munk formulae (22). In particular, new constants (α, γ, l) appear as compared with just r_∞ for Kubelka–Munk theory. However, this is an imaginary difficulty. As a matter of fact, both constants (α, γ, l) and r_∞ depend on just one parameter—the similarity parameter s (see above). This also means that additional parameters (α, γ, l) can be expressed via r_∞. However, the exact relationship between these parameters is more complicated as compared with initial suggestions of Kubelka and Munk (1931).

4. The accuracy of approximate analytical equations

The reflectance r calculated using equation (23) (accounting for equations (15)–(20)) and the radiative transfer code SCIATRAN (Rozanov *et al* 2005) is shown in figure 1 for different values of ω_0. Interestingly, the limit of a semi-infinite layer is approached already at $\tau = 5$ for the single scattering albedo 0.75 for the case studied (see figure 1).

It follows that the accuracy of equation (23) for r is high (see also figure 2). In particular, the relative error $\delta = 100(1 - r_{\rm ap}/r_{\rm ex})$ ($r_{\rm ap}$ is the approximate result and $r_{\rm ex}$ is the exact result) shown in figure 2 is below 5% at $\tau \geqslant 2$ and it is smaller than 2% at $\tau \geqslant 3$. Exact calculations (see tables 2 and 3) have been performed for several values of ω_0 using the phase function of the spherical polydispersions of

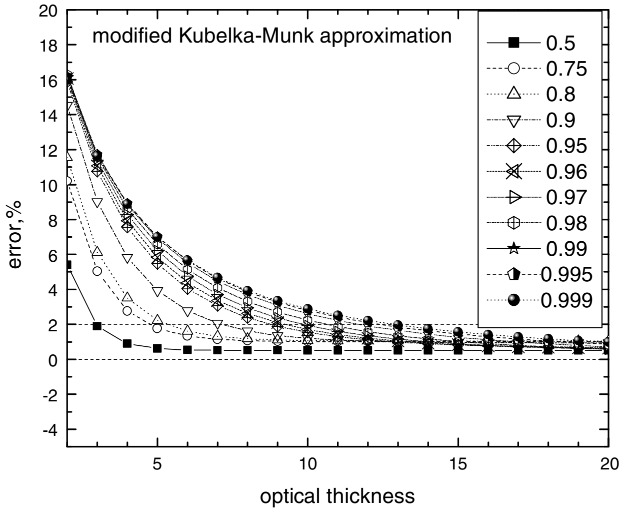

Figure 3. The same as in figure 2 except for MKMA.

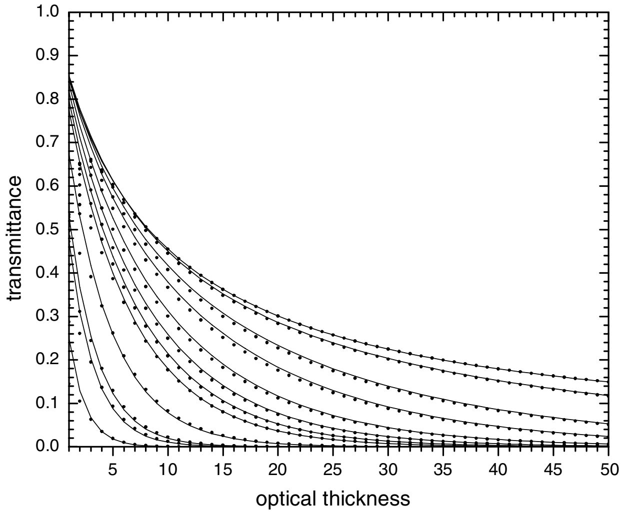

Figure 4. The same as in figure 1 except for the transmittance. Upper curves correspond to larger values of ω_0.

water droplets characterized by the gamma size distribution: $f(\Re) = B\Re^6 \exp(-1.5\Re)$, where B is the normalization constant and $\Re$ is the radius of droplets. The wavelength of 0.65 μm has been used for the Mie calculation of the phase function. Then the asymmetry parameter g is equal to 0.85.

The value of δ for the reflectance r calculated using equations (22) under assumption $\bar{x} \equiv x$ and r_∞ given by equation (19) has errors up to 16% at $\tau = 2$. We shall call equations (22) under the assumption $\bar{x} \equiv x$ and r_∞ given by equation (19) the modified Kubelka–Munk approximation (MKMA). The error of MKMA is below 2% for the reflectance at $\tau \geqslant 13$ (see figure 3). All parameters of MKMA have a clear meaning and can be calculated from first principles (e.g. s (see equation (20)), x). In particular, Mie theory can be used to derive the local optical characteristics $K_{\rm ext}$, ω_0, g.

The transmittance t calculated using asymptotic theory and the exact radiative transfer code SCIATRAN is shown in figure 4. It follows that there is a close correspondence between both calculations. The absolute value of the relative

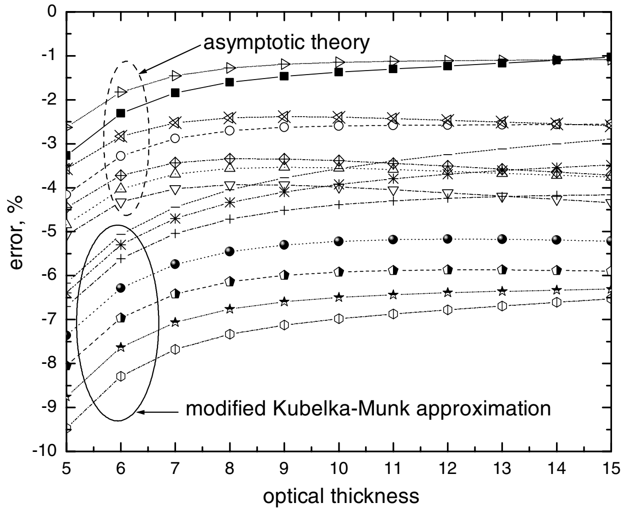

Figure 5. The same as in figures 2 and 3 except for the transmittance.

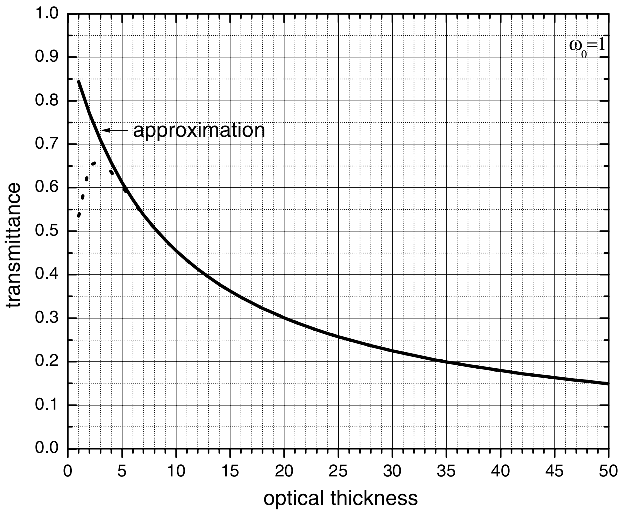

Figure 6. The transmittance calculated using the asymptotic theory (——) and the exact radiative transfer code (- - - -) at $\omega_0 = 1$, $g = 0.85$ for the polydispersion of spherical water droplets as discussed in the text.

error of asymptotic theory for the calculation of t (see equation (23)) is below 5% at $\tau \geqslant 5$ (see figure 5). It follows from figure 5 that the error of MKMA for the transmittance t is larger (up to -9.5% at $\tau = 5$). The comparison of figures 3 and 5 shows that generally the accuracy of approximations for r is higher than that for t. This is due to the fact that there is a peak in the dependence of t on τ obtained using numerical calculations with SCIATRAN (see figure 6), which is not captured by asymptotic theory. The physical reason behind the existence of such a peak is quite transparent. Indeed, t must decrease as $\tau \to \infty$. On the other hand, t represents the diffuse reflectance (the direct transmittance is excluded). Clearly, the diffuse transmittance must also vanish as $\tau \to 0$. This explains the origin of the peak in the dependence $t(\tau)$.

The absorptance $a = 1 - r - t$ calculated using asymptotic theory and also obtained from exact radiative transfer calculations is given in figure 7. There is good

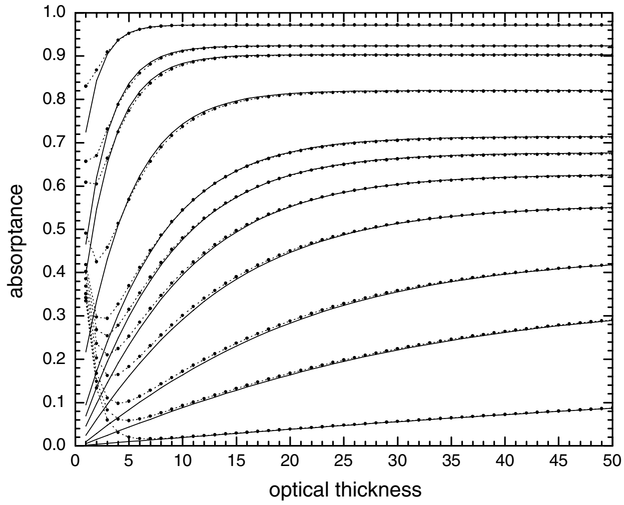

Figure 7. The same as in figure 1 except for the absorptance . Upper curves correspond to smaller values of ω_0.

correspondence between approximate and exact results at $\tau \geqslant 5$, where errors are below 5%. We found (not shown here) that MKMA can be applied for calculations of the absorptance with about the same accuracy as asymptotic theory. This is due to the cancellation of errors. Indeed, MKMA underestimates r (see figure 3) . However, it overestimates t (see figure 6) leading to quite reliable results for a.

5. Conclusions

The physical meaning of the Kubelka–Munk parameter $\sigma = \sqrt{K(K+S)}$ has been clarified. It coincides with the product of the diffusion exponent and the extinction coefficient of a light scattering material. The modified Kubelka–Munk approximation is introduced and its accuracy is checked against exact radiative transfer calculations at $g = 0.85$. The error of MKMA is smaller than 5% at $\tau \geqslant 7$ for r and a. It is better than 8% for the same range of τ for the transmittance. The error of asymptotic theory is smaller than 2% at $\tau \geqslant 3$ for r and 5% at $\tau \geqslant 5$ for t at $g = 0.85$. It was found that the use of asymptotic theory is preferable as compared with MKMA for thinner light scattering layers (but at $\tau \geqslant 5$).

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